APPM 4350 Final Project: Vibrations of a Cantilevered Beam

University of Colorado Boulder
Department of Applied Mathematics

AUTHOR INFORMATION
Aria Mundy[1]
Korben Smart[2]
Max Patwardhan[3]

[1]Applied Mathematics and Mechanical Engineering, SID: 108757957
[2]Engineering Physics, SID: 109859709
[3]Mechanical Engineering, SID: 109516516

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Abstract

The goal of this paper is to explore a modified version of the wave equation that describes the vibrations of a cantilever beam. While the general wave equation is useful in modeling the transverse motion of a stretched string or similar mediums that do not oppose bending, the modified beam-wave equation takes inertial properties along with resistance to bending into account. This paper will begin by applying the method of Separation of Variables to describe the motion of cantilever beam, and then apply the method of Variation of Parameters to describe the motion of the vibrating beam. An experiment to analyze the characteristics of the vibrating beam was executed, and these experimental results were compared to the results predicted by the derived models.

1 Introduction

1.1 Beam Deflection Background

Beam bending and vibration kinematics are an essential part of engineering analysis. The manner in which a beam bends under load is not at all intuitive, but using differential equations, the displacement of a deflecting beam can be calculated. A cantilever beam is a beam fixed at one end and subject to forces or moments anywhere along its body. The deflections of a vibrating a cantilever beam can be modeled with a modified version of the commonly known wave equation.

1.2 Cantilever Beam Vibration

This paper will attempt to solve a model which will measure the displacement of a vibrating cantilever beam at some given time. First, a homogeneous version of the problem will be solved using the method of Separation of Variables, and then the method of Variation of Parameters will be applied to account for a sinusoidal forcing function. An experiment will be carried out to analyze the behavior of the cantilever beam vibrations, and numerical results will be compared to data taken from a physical experiment. Lastly, some analysis of those results will be executed in order to determine certain properties regarding the harmonics of the cantilever beam.

2 Model Development

2.1 Beam Model

The motion of a stretched string can be modeled using the general wave equation of the following form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where c^2 represents the speed of wave propagation along the string and u represents the transverse displacement of the string from its stagnant position. While this equation is sufficient in describing the motion of a stretched string and other mediums that offer no resistance to bending, in order to analyze vibrations for mediums such as a stiff beam, Equation (1) must be further developed.

To analyze the vibrations of a stiff beam, the following model is used:

$$\rho \frac{\partial^2 u}{\partial t^2} = -Ek^2 \frac{\partial^4 u}{\partial s^4} \tag{2}$$

Where s is the position along the length of the beam, $u(s,t)$ is the transverse displacement of the beam, ρ is the density of the material, E is Young's Modulus of the material, and k is the radius of gyration of the cross-section.

Consider the case of a cantilever beam, in which a stiff beam is held rigidly at one end $s = 0$, and the beam is allowed to move freely at the other end $s = L$. Note that the Equation (1) contains a fourth-order spatial derivative. Thus, in order to solve this equation, four boundary conditions are necessary:

$$u(0,t) = 0, \tag{3a}$$

$$\frac{\partial}{\partial s} u(0,t) = 0, \tag{3b}$$

$$\frac{\partial^2}{\partial^2 s} u(L,t) = 0, \tag{3c}$$

$$\frac{\partial^3}{\partial^3 s} u(L,t) = 0. \tag{3d}$$

The support boundary conditions, as described by Equations (3a) and (3b), are types of Dirichlet boundary conditions, and are used to fix values of displacement and rotations on the boundary. Load and moment boundary conditions, as described by Equations (3c) and (3d), are types of Neumann boundary conditions, and reflect the fact that no external bending moment and no external force are applied to the free end of the beam; thus, the bending moment and shear force at the free end are also zero.

Note that the system defined by Equations (2) and (3) involves the motion of a beam that is fixed at one end an free to move at the other. The system that will ultimately be analyzed involves a beam that is being vibrated at a known frequency at one end, and free to move at the other.

2.2 Separation of Variables

Prior to analyzing the case of the beam that is vibrated at a known frequency, the method of Separation of Variables must be used to examine the simpler case in which the beam is fixed at one end. By applying Separation of Variables, it is possible to solve Equation (2) subject to the boundary conditions described in Equation (3).

It can be assumed that the displacement of the beam can be separated into two parts, one of which depends on position, and the other of which depends on time:

$$u(s,t) = F(s)G(t). \tag{4}$$

Substituting Equation (4) into the partial differential equation described in Equation (2) yields the following:

$$\frac{G''(t)}{G(t)} = \frac{-Ek^2}{\rho}\frac{F^{(4)}(s)}{F(s)}. \tag{5}$$

Since the left-hand side of Equation (5) does not change as s varies, it must evaluate to a constant. Using similar logic, since the right-hand side of Equation (5) does not change as t varies, it must also evaluate to a constant. Thus, Equation (5) can be set equal to some constant, λ:

$$\frac{G''(t)}{G(t)} = \frac{-Ek^2}{\rho}\frac{F^{(4)}(s)}{F(s)} = \lambda. \tag{6}$$

Taking λ to be a negative constant, we have the following equation for $G(t)$

$$G''(t) = -\lambda G(t) \tag{7}$$

as well as the following equation and boundary conditions for $F(s)$:

$$F(s) = \lambda F^{(4)}(s) \tag{8a}$$

$$F(0) = 0 \tag{8b}$$

$$F'(0) = 0 \tag{8c}$$

$$F''(L) = 0 \tag{8d}$$

$$F^{(3)}(L) = 0. \tag{8e}$$

A basis for the solutions of Equation (7) is $\left\{e^{i\sqrt{\lambda}t}, e^{-i\sqrt{\lambda}t}\right\}$, and using Euler's formula, this basis can be rewritten as $\left\{\cos(\sqrt{\lambda}t), \sin(\sqrt{\lambda}t)\right\}$. Thus , $G(t)$ satisfies the following:

$$G(t) = a\cos(\sqrt{\lambda}t) + b\sin(\sqrt{\lambda}t) \tag{9}$$

where a and b are constants.

Note, for mathematical convenience, β can be defined as follows:

$$\beta^2 = \frac{\rho\lambda}{Ek^2}. \tag{10}$$

A basis for the solutions of Equation (8) is $\left\{e^{\sqrt{\beta}s}, e^{\sqrt{\beta}s}, e^{i\sqrt{\beta}s}, , e^{i\sqrt{\beta}s}\right\}$, and using Euler's formula, this basis can be rewritten as $\left\{\cos(\sqrt{\beta}s), \sin(\sqrt{\beta}s), \sinh(\sqrt{\beta}s), \cosh(\sqrt{\beta}s)\right\}$. Thus, $F(s)$ satisfies the following:

$$F(s) = c_1\cos(\sqrt{\beta}s) + c_2\sin(\sqrt{\beta}s) + c_3\cosh(\sqrt{\beta}s) + c_4\sinh(\sqrt{\beta}s). \tag{11}$$

Applying the boundary conditions defined in Equation (8) to Equation (11) yields the following system of equations that the coefficients must satisfy:

$$\begin{bmatrix} \cosh(\sqrt{\beta}L) + \cos(\sqrt{\beta}L) & \sinh(\sqrt{\beta}L) + \sin(\sqrt{\beta}L) \\ \sinh(\sqrt{\beta}L) - \sin(\sqrt{\beta}L) & \cosh(\sqrt{\beta}L) + \cos(\sqrt{\beta}L) \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

Setting the determinant of the matrix equal to zero, a condition that the nontrivial solutions must satisfy is obtained:

$$\begin{vmatrix} \cosh(\sqrt{\beta}L) + \cos(\sqrt{\beta}L) & \sinh(\sqrt{\beta}L) + \sin(\sqrt{\beta}L) \\ \sinh(\sqrt{\beta}L) - \sin(\sqrt{\beta}L) & \cosh(\sqrt{\beta}L) + \cos(\sqrt{\beta}L) \end{vmatrix} = 0. \tag{13}$$

Evaluation of Equation (13) yields the following frequency equation for the cantilevered beam:

$$\cosh(\sqrt{\beta_n}L)\cos(\sqrt{\beta_n}L) = -1. \tag{14}$$

Note that the n subscript is included because there are multiple roots which satisfy Equation (14). Equation (14) can be solved numerically for the constants, $\sqrt{\beta_n}L$, and these constants, along with Equation (10), can be used to find the natural frequencies of the cantilever beam.

Equation (11) can be simplified to the following form:

$$\begin{aligned} F_n(s) = c_4 \Bigg[&\left(\sinh(\sqrt{\beta_n}s) - \sin(\sqrt{\beta_n}s)\right) \\ &+ \left(\frac{\cos(\sqrt{\beta_n}L) + \cosh(\sqrt{\beta_n}L)}{\sin(\sqrt{\beta_n}L) - \sinh(\sqrt{\beta_n}L)}\right)\left(\cosh(\sqrt{\beta_n}s) - \cos(\sqrt{\beta_n}s)\right) \end{aligned} \tag{15}$$

where the unknown constant c_4, which is oftentimes complex, can be determined using the initial conditions of the beam (in other words, the displacements and velocity of the beam at the instant $t = 0$).

Referring back to Equation (4), the spatial solution and time-domain solution can be combined to describe the beam vibration in the final step of Separation of Variables:

$$u_n(x,t) = F_n(s)\left[a_n\sin(\sqrt{\lambda_n}t) + b_n\cos(\sqrt{\lambda_n}t)\right] \tag{16}$$

where $F_n(s)$ is given by Equation (15) and λ_n satisfies Equations (10) and (14).

Applying the Superposition Principle, the solution of the Equation (2) is the linear combination of the product solutions $u_n(x,t)$:

$$u(s,t) = \sum_{n=1}^{\infty} F_n(s) \left[a_n \sin(\sqrt{\lambda_n} t) + b_n \cos(\sqrt{\lambda_n} t) \right]. \tag{17}$$

Thus, through the method of Separation of Variable, a model that describes the homogeneous system is obtained.

2.3 Variation of Parameters

In order to solve the wave equation with known forcing, the method of Variation of Parameters will be applied to the results found in the previous section. Variation of Parameters is a method used to solved differential equations with known forcing. Since the partial differential equation in Equation (2) is second-order with respect to time, it can be regarded similarly to a second-order, linear ordinary differential equation with constant coefficients and a time-dependent forcing function.

Variation of Parameters can be executed in the following manner: once the homogeneous solution is known (note that the homogeneous solution must contains two linearly independent solutions to the homogeneous problem, given that the differential equation is second-order with respect to time), it can be assumed that the particular solutions is of the same form as the homogeneous solution, except for the fact that the coefficients now depend on time. Two equations are needed to solve for the time-dependent coefficients. The first equation can be obtained by noting that the proposed solution must satisfy the differential equation. The second equation can come from a variety of places, but in this case, the second equation will come from a clever assumption that will greatly simplify the calculations involved in this method. Once the two equations are obtained, the time-dependent coefficients can be determined, and the particular solution can be determined.

Referring to the boundary conditions defined in Equation (3), it is now assumed that at $s = 0$, the beam is being vibrated at a known frequency. Thus, the boundary conditions can be written as follows:

$$u(0,t) = A \sin(\Omega t), \tag{18a}$$

$$\frac{\partial}{\partial s} u(0,t) = 0, \tag{18b}$$

$$\frac{\partial^2}{\partial^2 s} u(L,t) = 0, \tag{18c}$$

$$\frac{\partial^3}{\partial^3 s} u(L,t) = 0. \tag{18d}$$

The initial value problem and the boundary conditions can be redefined such that the forcing term is moved to the initial value problem, and the resultant boundary conditions form a vector space. It can be assumed that the initial value problem is of the form

$$u(s,t) = w(s,t) + A\sin(\Omega t)\gamma(s) \tag{19}$$

where $\gamma(s)$ is an unknown function of s. However, since the forcing term, $A\sin(\Omega t)$ is independent of s, this implies that $\gamma(s) = 1$. Substituting Equation (19) into Equation (2) yields the following:

$$\frac{\partial^2 w}{\partial t^2} - A\Omega^2 \sin(\Omega t) = -\left(\frac{EI}{\rho A}\right)^2 \frac{\partial^4 u}{\partial s^4} \tag{20}$$

where I is the moment of inertia and A is the cross sectional area, with the following boundary conditions:

$$w(0,t) = 0, \tag{21a}$$

$$\frac{\partial}{\partial s} w(0,t) = 0, \tag{21b}$$

$$\frac{\partial^2}{\partial^2 s} w(L,t) = 0, \tag{21c}$$

$$\frac{\partial^3}{\partial^3 s} w(L,t) = 0. \tag{21d}$$

As outlined in the method of Variation of Parameters, the particular solution with the forcing term will take the same form as Equation (17), except a_n and b_n are now assumed to be time-dependent:

$$u(s,t) = \sum_{n=1}^{\infty} F_n(s)\left[a_n(t)\sin(q_n t) + b_n(t)\cos(q_n t)\right]. \tag{22}$$

Note that the $\sqrt{\lambda_n}$ terms have been replaced with q_n for mathematical convenience. Recall from the previous section that allowable values for q_n are found from Equation (14), where each n corresponds to an allowable value of q_n. For all allowable values of q_n, $F_n(s)$ can be determined using Equation (15).

Applying the method of Variation of Parameters, recall that two equations are needed to solve for the time-dependent coefficients, where the first equation can be obtained by differentiating Equation (22), and substituting the results back into the partial differential equation described by Equation (20). It can be shown that

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} F_n(s)\left[q_n a_n(t)\cos(q_n t) + a_n'(t)\sin(q_n t) - q_n b_n(t)\sin(q_n t) + b_n'(t)\cos(q_n t)\right]. \tag{23}$$

Now, a clever assumption can be made, which not only provides the second equation necessary to solve for the coefficients, but also greatly simplifies the calculations for the second derivative calculation that is needed to determine the first equation. As outlined in the method of Variation of Parameters, one can assume that the second equation is of the form

$$a_n'(t)u_1(t) + b_n'(t)u_2(t) = 0 \tag{24}$$

where $u_1(t)$ and $u_2(t)$ are solutions to the homogeneous problem. In this case, $u_1(t) = \sin(q_n t)$ and $u_2(t) = \cos(q_n t)$ so

$$a'(t)\sin(q_n t) + b'(t)\cos(q_n t) = 0 \tag{25}$$

With this assumption, the second time derivative is found to be

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} F_n(s)\left[-q_n^2 a_n(t)\sin(q_n t) + q_n a_n'(t)\cos(q_n t) - q_n^2 b_n(t)\cos(q_n t) - q_n b_n'(t)\sin(q_n t)\right]. \tag{26}$$

Finally, the fourth order spatial derivative is

$$\frac{\partial^4 u}{\partial s^4} = \sum_{n=1}^{\infty} F_n^4(s)\left[a_n(t)\cos(q_n t) + b_n(t)\sin(q_n t)\right]. \tag{27}$$

Substituting Equations (26) and (27) into Equation (20) yields the following result:

$$\begin{aligned}\sum_{n=1}^{\infty} F_n(s)\left[-q_n^2\left(a_n(t)\sin(q_n t) + b_n(t)\cos(q_n t)\right) + q_n\left(a_n'(t)\cos(q_n t) - b_n'(t)\sin(q_n t)\right)\right]\\ + \left(\frac{EI}{\rho A}\right)^2\left[\sum_{n=1}^{\infty} F_n^4(s)\left[a_n(t)\cos(q_n t) + b_n(t)\sin(q_n t)\right]\right] = A\Omega^2\sin(\Omega t).\end{aligned} \tag{28}$$

Recall that $a_n(t)\sin(q_n t) + b_n(t)\cos(q_n t) = 0$ is the solution to the homogeneous equation, so Equation (28) simplifies to

$$\sum_{n=1}^{\infty} F_n(s)\left[q_n\left(a_n'(t)\cos(q_n t) - b_n'(t)\sin(q_n t)\right)\right] = A\Omega^2\sin(\Omega t). \tag{29}$$

Now, it can be shown that for each allowable value of q, the coefficients of $F_n(s)$ are orthogonal. Note that $F_n(s)$ satisfies the equation

$$\frac{\mathrm{d}^4}{\mathrm{d}s^4}F_n(s) = q_n^4 F_n(s) \tag{30}$$

since the derivatives of the cosine and sine functions (as well as hyperbolic cosine and hyperbolic sine functions) repeat every fourth cycle. This equation is subject to the same boundary conditions given in Equation (8). Two arbitrary eigenvalues and associated eigenfunctions are chosen, $\{F_n(s), q_n\}$ and $\{F_m(s), q_m\}$, both of which solve Equation (30), implying that

$$\frac{\mathrm{d}^4}{\mathrm{d}s^4}F_n(s) = q_n^4 F_n(s) \tag{31a}$$

$$\frac{\mathrm{d}^4}{\mathrm{d}s^4}F_m(s) = q_m^4 F_m(s). \tag{31b}$$

Multiplying Equation (31a) by $F_m(s)$ and Equation (31b) by $F_n(s)$, and integrating by parts twice gives the following:

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}s^4}F_n(s)F_m(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int \frac{\mathrm{d}^4}{\mathrm{d}s^4}F_n(s)F_m(s)ds = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^3}{\mathrm{d}s^3}F_n(s)F_m(s) - \int \frac{\mathrm{d}^3}{\mathrm{d}s^3}F_n(s)\frac{\mathrm{d}}{\mathrm{d}s}F_m(s)ds\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^3}{\mathrm{d}s^3}F_n(s)F_m(s) - \frac{\mathrm{d}^2}{\mathrm{d}s^2}F_n(s)\frac{\mathrm{d}}{\mathrm{d}s}F_m(s) + \int \frac{\mathrm{d}^2}{\mathrm{d}s^2}F_n(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}F_m(s)ds\right]\end{aligned} \tag{32a}$$

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}s^4}F_m(s)F_n(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int \frac{\mathrm{d}^4}{\mathrm{d}s^4}F_m(s)F_n(s)ds = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^3}{\mathrm{d}s^3}F_m(s)F_n(s) - \int \frac{\mathrm{d}^3}{\mathrm{d}s^3}F_m(s)\frac{\mathrm{d}}{\mathrm{d}s}F_n(s)ds\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^3}{\mathrm{d}s^3}F_m(s)F_n(s) - \frac{\mathrm{d}^2}{\mathrm{d}s^2}F_m(s)\frac{\mathrm{d}}{\mathrm{d}s}F_n(s) + \int \frac{\mathrm{d}^2}{\mathrm{d}s^2}F_m(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}F_n(s)ds\right]\end{aligned} \tag{32b}$$

Equation (32a) and (32b) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^3}{\mathrm{d}s^3}F_n(s)F_m(s)\right] - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}F_n(s)\frac{\mathrm{d}}{\mathrm{d}s}F_m(s)\right] + \frac{\mathrm{d}^2}{\mathrm{d}s^2}F_n(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}F_m(s)ds = q_n^4 F_m(s)F_n(s) \tag{33a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^3}{\mathrm{d}s^3}F_m(s)F_n(s)\right] - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2}F_m(s)\frac{\mathrm{d}}{\mathrm{d}s}F_n(s)\right] + \frac{\mathrm{d}^2}{\mathrm{d}s^2}F_m(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2}F_n(s)ds = q_m^4 F_n(s)F_m(s) \tag{33b}$$

Subtracting Equation (33a) from Equation (33b), integrating over $0 < s < L$, and applying the boundary conditions from Equation (8) gives the following result:

$$(q_m^4 - q_n^4)\int_0^L F_m(s)F_n(s)ds = 0. \tag{34}$$

However, since q_m and q_n are unique eigenvalues, the integral must evaluate to zero for Equation (34) to be true. Thus, by the definition of orthogonality, every $F_n(s)$ is orthogonal.

Now, Equation (29) can be rewritten such that a $\sum_{n=1}^{\infty} F_n(s)$ term can be factored out on both sides:

$$\sum_{n=1}^{\infty} F_n(s)\left[q_n\left(a_n'(t)\cos(q_n t) - b_n'(t)\sin(q_n t)\right)\right] = \sum_{n=1}^{\infty} B_n F_n(s)\left[A\Omega^2\sin(\Omega t)\right] \tag{35}$$

in which

$$\sum_{n=1}^{\infty} B_n F_n(s) = 1 \tag{36}$$

Equation (35) can now be re-written as

$$\sum_{n=1}^{\infty} F_n(s) \left[q_n \left(a_n'(t)\cos(q_n t) - b_n'(t)\sin(q_n t)\right) - B_n F_n(s) \left(A\Omega^2 \sin(\Omega t)\right)\right] = 0 \tag{37}$$

Once again, orthogonality is utilized. By taking the inner product of both sides of Equation (35) with F_j, recalling that the F_n terms are orthogonal, the only term that survives is when n = j. Therefore,

$$|F_j(s)| \left[q_j \left(a_j'(t)\cos(q_j t) - b_j'(t)\sin(q_j t)\right) - B_j F_j(s) \left(A\Omega^2 \sin(\Omega t)\right)\right] = 0 \tag{38}$$

In addition, the orthogonality of the F_n terms can be employed to find an equation for B_n. Multiplying both sides of Equation (36) by F_j and integrating from $0 < s < L$ yields the following equation that B_n must satisfy:

$$B_j \int_0^L F_j^2(s)ds = \int_0^L F_j(s)ds \tag{39}$$

Equations (25), (38), and (39) desribe a system of three equations containing three unknowns: $a_n'(t)$, $b_n'(t)$, and B_n. Solving this system will yield solutions to the desired coefficients. The coefficients $a_n(t)$ and $b_n(t)$ can be solved for using Equations (25) and (38), and applying Cramer's rule. If the determinants D, D_a, and D_b are defined as

$$D = \begin{vmatrix} \sin(q_n t) & \cos(q_n t) \\ q_n t\cos(q_n t) & q_n t\sin(q_n t) \end{vmatrix} = -q_n \tag{40a}$$

$$D_a = \begin{vmatrix} AB_n\Omega^2\sin(\Omega t) & \cos(q_n t) \\ 0 & q_n t\sin(q_n t) \end{vmatrix} = AB_n q_n \Omega^2 \sin(\Omega t)\sin(q_n t) \tag{40b}$$

$$D_b = \begin{vmatrix} \sin(q_n t) & AB_n\Omega^2\sin(\Omega t) \\ q_n t\cos(q_n t) & 0 \end{vmatrix} = -AB_n q_n \Omega^2 \sin(\Omega t)\cos(q_n t) \tag{40c}$$

then, as outlined in Cramer's rule, $a_n'(t)$ and $b_n'(t)$ are equivalent to the following:

$$a_n'(t) = \frac{D_a}{D} = -AB_n\Omega^2\sin(\Omega t)\sin(q_n t) \tag{41a}$$

$$b_n'(t) = \frac{D_b}{D} = AB_n\Omega^2\sin(\Omega t)\cos(q_n t). \tag{41b}$$

Software such as Mathematical can be used to integrate Equation (41), which yields the following result for $a_n(t)$ and $b_n(t)$:

$$a_n(t) = \frac{AB_n\Omega^2\left(\sin(q_n t)\sin(\Omega t)\cos(\Omega t) - \Omega\sin(q_n t)\cos(\Omega t)\right)}{(q_n - \Omega)(q_n + \Omega)} \tag{42a}$$

$$b_n(t) = \frac{AB_n\Omega^2\left(q_n\sin(q_nt)\sin(\Omega t) + \Omega\cos(q_nt)\cos(\Omega t)\right)}{(q_n-\Omega)(q_n+\Omega)} \tag{42b}$$

Finally, B_n can be evaluated by integrating Equation (39).

Thus, through the method of Variation of Parameters, a model of the physical system involving a cantilever beam that is vibrated at a known frequency is obtained.

3 Numerical Experiment

3.1 Experiment Overview

Each of the exact solutions to Equation (2) is called a "Fourier mode," and the solution to the initial value problem can be constructed by summing over these "Fourier modes." Each "Fourier mode" has a corresponding frequency of vibration and spatial structure, which can be experimentally explored. The purpose of this experiment was to show that the "Fourier modes" of each strip correspond with natural frequencies, as defined by Equation (14).

To explore these spatial structures at specific frequencies, an experiment was carried out in which resonant strips of different were vibrated at fixed frequencies and the spatial structures were observed. In order to do this, a waveform generator was attached to to a Mechanical Wave Driver via BNC and banana cables. A collection of resonance strips were attached to the wave driver which were then vibrated at a voltage with varying input frequencies. While voltage of 2V was initially used, it was eventually increased to 6V, as the vibrations were significantly more pronounced.

The input frequency was initially set to 20Hz, and then slowly incremented to determine harmonic frequencies. Once a particular beam was at a harmonic, a strobe light was tuned to flash at an identical frequency, making the beam appear as if it had been frozen in place. By slightly varying the frequency of the strobe light while the beam was at a harmonic frequency, the beam appeared to move in slow-motion. In an effort to preserve the equipment, the frequency was kept between 20Hz and 180Hz, as anything outside of this range could damage either the mechanical wave driver and resonance strips.

The fundamental frequencies that characterized the first mode of vibration were found for all six strips. Then, the harmonic frequencies that characterized the second mode of vibration were found for the longest three strips. Finally, the harmonic frequency that characterized the third mode of vibration was found for the longest strip. For the harmonic frequencies that were found, the distance between the nodes was also recorded.

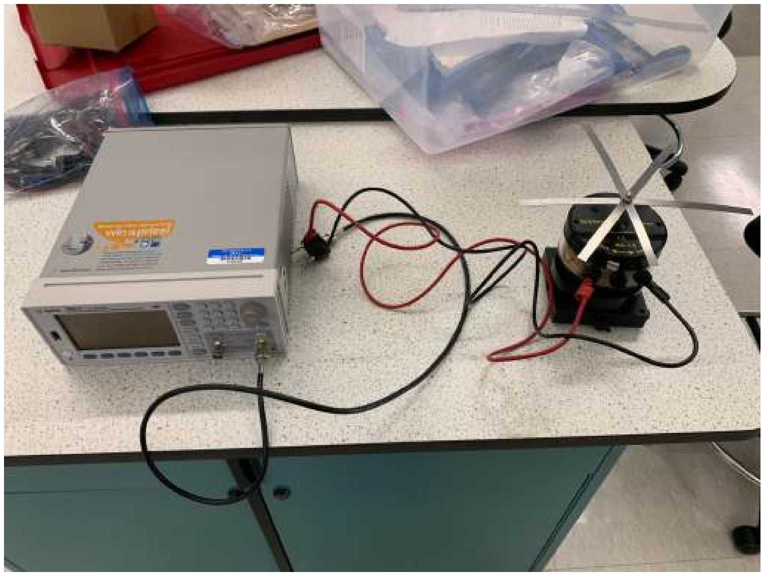

Figure 1: Experimental setup with waveform generator and mechanical wave driver (with resonance strips) attached via BNC cable.

3.2 Determining the Harmonic Frequencies

To numerically determine the harmonic frequencies of the resonant strips used in this experiment, Equation (14) can be applied. As mentioned earlier, for each value of L in Equation (14), there are multiple roots that satisfy the equation.

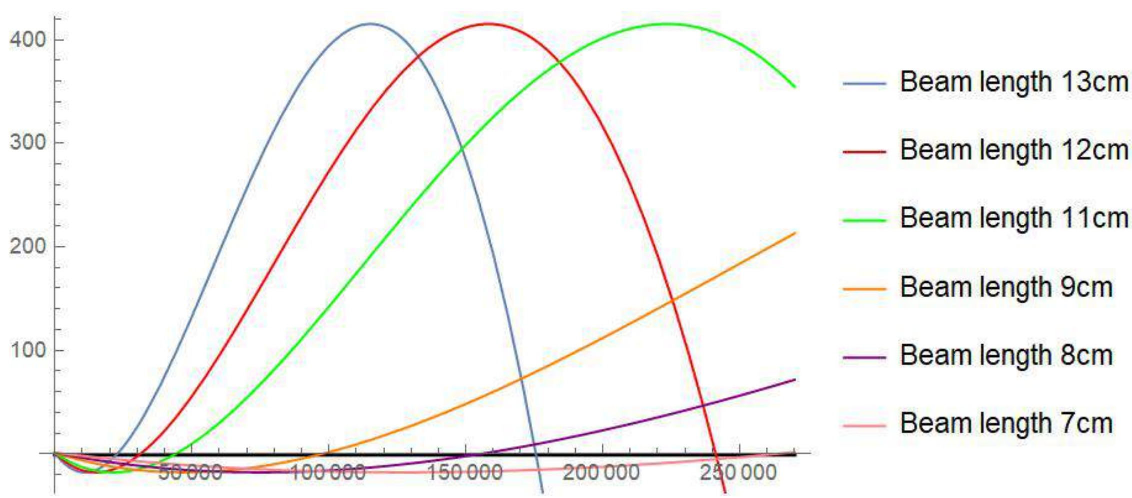

Figure 2: Solutions of Equation (14) for various values of L.

For these various values of L, Equation (14) can be solved, which yields the following values for λ :

Length(cm)	$\lambda 1$	$\lambda 2$	$\lambda 3$
13.1	570.4	22,219.6	174,205.5
12.1	783.6	30,526.8	239,334.7
11.1	1,106.5	43,105.3	N/A
9.1	2,429.7	95,423.8	N/A
8.1	3,870.7	152,013.0	N/A
7.1	6,556.7	257,506.8	N/A

Table 1: Solutions of Equation (14) for various values of L.

Upon initial analysis, these values appear unreasonably high. To fix this, dimensional analysis can be utilized to find the units of λ. Equation (14) must be unitless, which implies that $\sqrt{\beta_n}L$ is unitless as well. If Equation (9) is substituted into this, the following equation is obtained. Note that

$$\sqrt{\beta_n}L = \sqrt[4]{\frac{\rho\lambda}{Ek^2}} \cdot L$$

where substituting units yields the following result:

$$\sqrt[4]{\frac{kg}{m^3} \cdot \lambda \cdot \frac{m \cdot s^2}{kg} \cdot \frac{1}{m^2}} = \sqrt[4]{\lambda \cdot s^2}$$

Since the above expression must be unitless, it is clear that the units on λ must be $\frac{1}{s^2}$ or Hz^2. This means that taking the square roots of the values found before will provide the true harmonic frequencies for each strip. The frequencies were found as follows:

Length(cm)	$\omega 1$	$\omega 2$	$\omega 3$
13.1	23.9	149.1	417.4
12.1	27.9	174.7	489.2
11.1	33.3	207.6	NA
9.1	49.2	308.9	NA
8.1	62.2	389.9	NA
7.1	81.0	507.5	NA

Table 2: Harmonic frequencies for various values of L.

The position of the nodes at each harmonic frequency is dependent on which harmonic frequency the beam is vibrating at. At the first harmonic frequency, the node will sit at $s = L$. For the second frequency, there should be two nodes at $s = L$ and $s = \frac{L}{2}$. For the third frequency, there should be three nodes at $s = L$, $s = \frac{2L}{3}$, and $s = \frac{L}{3}$. The position of the nodes can be summarized as follows:

Length (cm)	node position (cm), $\omega1$	node position (cm), $\omega2$	node position (cm), $\omega3$
13.1	13.1	6.6, 13.1	4.4, 8.7, 13.1
12.1	12.1	6.1, 12.1	4.0, 8.1, 12.1
11.1	11.1	5.6, 11.1	3.7, 7.4, 11.1
9.1	9.1	4.6, 9.1	3.0, 6.1, 9.1
8.1	8.1	4.1, 8.1	2.7, 5.4, 8.1
7.1	7.1	3.6, 7.1	2.3, 4.7, 7.1

Table 3: Node positions for harmonic frequencies for various values of L.

3.3 Relationship Between Harmonic Frequencies

Modern musical instruments are tuned such that there is perfect symmetry an octave apart. Take for example that a middle A-note is tuned to vibrate at 440Hz. An A-note one octave above it will have a frequency of 880Hz, similarly an A-note one octave below will have a frequency of 220Hz. In Western Music Theory, there are twelve notes, and on a stringed instrument, each note is tuned to be an equal step from the last in each octave. This means that each harmonic frequency occurs at a frequency that is $\sqrt{12}$ times larger than the previous one. This causes stringed instruments to be melodic. If the ratio between the first two harmonic frequencies for any of the beams is compared, it can be shown that the second harmonic frequency is around 6.2 times larger than the first harmonic frequency. If the harmonic frequencies of the cantilevered beams are rationally related, it would be expected to find the third harmonic frequency 6.2 times higher than the second. However, the calculated ratio between the second and third harmonic frequencies of all the beams is found to be around 2.8. This change indicates that there is not a constant rational relationship between each harmonic mode for a stiff beam.

3.4 Comparison of Experimental and Analytic results

During the experiment the following data were collected:

Length(cm)	$\omega1$	$\omega2$	$\omega3$
13.1	21.4	138	N/A.
12.1	24.5	165	N/A
11.1	35	179	N/A
9.1	44	N/A	N/A
8.1	56	N/A	N/A
7.1	69	N/A	N/A

Table 4: Experimentally determined harmonic frequencies for various values of L.

The values collected during the experiment were consistently below the values predicted by the analysis, and the error of these measurements ranged from 10% to 15%. This error can likely be attributed to the experimental setup. Note that all six beams were connected during the experiment. In addition, at the beginning of the experiment, each beam was equally spaced, but as the data was being collected, the vibrations caused the beams to move and distorted the spacing between each beam. These factors, combined with the loose fastening of the beam to the vibrating pin, would likely cause some damping in the vibrations of each beam. While there are other potential factors that could be the cause of the discrepancy, these sources of dampening are the likely reason the results were below the predicted values. Interestingly, there was one single measurement where the measured value was actually larger than the predicted value, which occurred for the first harmonic frequency of the 11 centimeter beam.

The measured positions of the nodes for each harmonic frequency are summarized in the following table:

Length (cm)	node position (cm), $\omega 1$	node position (cm), $\omega 2$	node position (cm), $\omega 3$
13.1	13.1	6.4, 13.1	N/A
12.1	12.1	6.2, 12.1	N/A
11.1	11.1	5.5, 11.1	N/A
9.1	9.1	N/A	N/A
8.1	8.1	N/A	N/A
7.1	7.1	N/A	N/A

Table 5: Experimentally determined node positions for harmonic frequencies for various values of L.

The predicted and experimentally determined values are quite similar. However, it was not possible to experimentally determine the node positions of the second harmonic frequencies for the beams that were 9 centimeters or shorter, nor any frequencies for the third harmonic, as it was decided that trying to test those frequencies could have resulted in damage to the experimental equipment.

3.5 The Speed of Waves Through a Beam

The speed of a wave through a stretched string is defined as $\frac{\omega}{k}$, where ω is the angular frequency of the wave and k is the wave number. For a stiff beam, this equation becomes more complicated. To determine the speed of waves through a beam, an equivalent statement for ω must be determined. This can be done using the following equation:

$$\beta = \frac{\lambda \rho}{E k^2}. \tag{43}$$

Recall from Equation (47) that λ is in units of Hz^2, let $\lambda = \omega^2$, and substitute into Equation (46):

$$\beta = \frac{\omega^2 \rho}{E k^2}. \tag{44}$$

Now solving this equation for ω yields:

$$\omega = \pm \beta k \sqrt{\frac{E}{\rho}} \tag{45}$$

Because the speed of the wave is being calculated, the absolute value of ω can be used. For the motion of waves in a stiff beam, it is clear to see that $\sqrt{\beta}$ is equivalent to the wave number in a stiff beam. Dividing Equation (45) by this value yields:

$$\frac{\omega}{\sqrt{\beta}} = k \sqrt{\frac{E\beta}{\rho}} \tag{46}$$

This equation is the speed of wave propagation through a stiff beam. It is evident that the wave propagation depends on β, which changes depending on how fast the beam is being vibrated.

4 Conclusion

4.1 Summary of Results

In this paper, a partial differential equation was introduced to extend the wave equation and describe the vibrations of stiff beam. By using the method of Separation of Variables, a spatial domain equation and time domain equation were determined, and combined to ultimately determine a solution describing the transverse displacement of the beam for the homogeneous case, where the beam is fixed at one end and free to move at the other end.

Next, the boundary conditions were modified to account for the fact that the beam is vibrated at a known frequency at the fixed end and free to move at the other end. To solve this partial differential equation with a time-dependent forcing function, the method of Variation of Parameters was applied. A solution describing the transverse displacement of the vibrating beam, including equations that describe the time-dependent coefficients of this solution, were found.

Finally, an experiment was carried out, where a series variable-length cantilever beams were oscillated at various frequencies. Equation (14) was solved in order to find the expected frequencies for which harmonics occur. From this, the experimental frequencies could be compared against expected frequencies. It was shown that the experimental frequencies were 10-15% lower than what was numerically calculated. There are many potential reasons for this, the most likely of which was a partial damping effect at the base of the resonance strips.

In addition, it was shown that there is no rational step between harmonic frequencies of strings, rather there exists an ever-changing ratio between each fundamental harmonic.

4.2 Future Research

In future it might prove interesting to study the motion of beams with different forcing functions, changing the boundary condition for solving with variation of parameters. Another interesting thing to explore could be changing the shape of the beam; potentially studying something with a circular or triangular cross section.